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Non-static spherically symmetric charged dust distribution in general relativity

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Abstract. This note establishes two theorems for a spherically symmetric charged dust distribution. The first theorem states that the expansion (or contraction) of a spherically symmetric charged dust cannot be isotropic and the second theorem states that a static charge is unstable to perturbations, which do not destroy the spherical symmetry, and would eventually collapse.

1. Introduction

In recent years, the dynamics of a charged dust distribution has attracted some attention (Bondi and Lyttleton 1959, Liboff 1966). In the present note we set up the Einstein–Maxwell equations for a non-static spherically symmetric charged dust distribution and find that any expansion (or contraction) must be accompanied by shear. Further, it is shown that a spherically symmetric equilibrium distribution (i.e. one in which the charge density σ is equal to the matter density ρ) would eventually collapse, if it were thrown out of the static state without destroying spherical symmetry.

2. Field equations

The general spherically symmetric line element may be taken as

$$ds^2 = e^{\nu(r,t)}\,dt^2 - e^{\lambda(r,t)}\,dr^2 - e^{\omega(r,t)}(d\theta^2 + \sin^2\theta\,d\phi^2). \qquad (1)$$

Here we assume the coordinate system to be co-moving, so that

$$v^\alpha = \frac{\delta_4^\alpha}{\sqrt{g_{44}}}.$$

The Einstein–Maxwell field equations in usual notations are

$$R_\beta^\alpha - \tfrac{1}{2}\delta_\beta^\alpha R = -8\pi(T_\beta^\alpha + M_\beta^\alpha) \qquad (2)$$

with

$$T_\beta^\alpha = \frac{1}{16\pi}\delta_\beta^\alpha(F^{ab}F_{ab}) - \frac{1}{4\pi}F^{\alpha a}F_{\beta a}$$

and

$$M_\beta^\alpha = \rho v^\alpha v_\beta$$

$$F^{\alpha\beta}_{;\beta} = 4\pi J^\alpha \qquad (3)$$

with

$$J^\alpha = \sigma v^\alpha \qquad (4)$$

$$F_{[\alpha\beta,\gamma]} = 0 \qquad (5)$$

where σ is the charge density and ρ the matter density. We shall assume that the dust is uniformly charged, i.e. σ/ρ is independent of r. As is well known, it follows from the field equations that σ/ρ remains constant along stream lines which are here the t lines. Hence in this case σ/ρ is a constant. The possible non-vanishing components of $F_{\mu\nu}$ as consistent with spherical symmetry, are only F_{14} $(= -F_{41})$ and F_{23} $(= -F_{32})$. If we take the divergence of equation (2), we get

$$F_{14} = \frac{\rho}{\sigma}\frac{\nu'}{2}e^{\nu/2}. \qquad (6)$$

Also from (3) with the line element (1), we obtain

$$\frac{\partial F_{14}}{\partial t} + \left(\dot{\omega} - \frac{\dot{\lambda}+\dot{\nu}}{2}\right) F_{14} = 0 \tag{7}$$

$$\left\{\frac{\partial F_{14}}{\partial r} + \left(\omega' - \frac{\lambda'+\nu'}{2}\right) F_{14}\right\} e^{-(\lambda+\nu)} = 4\pi J^4. \tag{8}$$

Further, from (5), $F_{23,1} = 0$, $F_{23} = A$, where A is at most a function of t alone. In the expression of T^{α}_{β} we would have, consequently, the term $F_{23}F^{23} = e^{-2\omega}(\sin^2\theta)^{-1}A^2$. For the line element (1) to be regular at the origin with θ, ϕ as angular coordinates, e^{ω} must tend towards zero as r^2 as $r \to 0$. Thus T^{α}_{β} would become arbitrarily large as $r \to 0$. Hence, if the solution is to be regular at the origin, one must have $F_{23} = 0$.

Equation (2) written out explicitly gives

$$F^{14}F_{14} = -e^{-\omega} + e^{-\lambda}(\tfrac{1}{2}\omega'\nu' + \tfrac{1}{4}\omega'^2) + e^{-\nu}(-\tfrac{3}{4}\dot{\omega}^2 + \tfrac{1}{2}\dot{\omega}\dot{\nu} - \ddot{\omega}) \tag{9}$$

$$\begin{aligned} -F^{14}F_{14} = {} & e^{-\lambda}(\tfrac{1}{2}\omega'' + \tfrac{1}{4}\omega'^2 + \tfrac{1}{2}\nu'' + \tfrac{1}{4}\nu'^2 - \tfrac{1}{4}\omega'\lambda' + \tfrac{1}{4}\omega'\nu' - \tfrac{1}{4}\nu'\lambda') \\ & + e^{-\nu}(-\tfrac{1}{2}\ddot{\lambda} - \tfrac{1}{4}\dot{\lambda}^2 - \tfrac{1}{2}\ddot{\omega} - \tfrac{1}{4}\dot{\omega}^2 + \tfrac{1}{4}\dot{\lambda}\dot{\nu} + \tfrac{1}{4}\dot{\omega}\dot{\nu} - \tfrac{1}{4}\dot{\omega}\dot{\lambda}) \end{aligned} \tag{10}$$

$$-8\pi\rho + F^{14}F_{14} = -e^{-\omega} + e^{-\lambda}(\omega'' + \tfrac{3}{4}\omega'^2 - \tfrac{1}{2}\omega'\lambda') - e^{-\nu}(\tfrac{1}{2}\dot{\lambda}\dot{\omega} + \tfrac{1}{4}\dot{\omega}^2) \tag{11}$$

and

$$0 = e^{-\nu}(-\dot{\omega}' - \tfrac{1}{2}\dot{\omega}\omega' + \tfrac{1}{2}\dot{\lambda}\omega' + \tfrac{1}{2}\dot{\omega}\nu'). \tag{12}$$

From equations (6) and (7) it follows that

$$\dot{\nu}' + \nu'(\dot{\omega} - \tfrac{1}{2}\dot{\lambda}) = 0. \tag{13}$$

Equations (3) and (4) give the charge density

$$\begin{aligned} \sigma &= |J_i J^i|^{1/2} \\ &= \frac{1}{8\pi}\frac{\rho}{\sigma} e^{-\lambda}\{\nu'' + \nu'(\omega' - \tfrac{1}{2}\lambda')\} \end{aligned}$$

so that

$$\frac{8\pi\sigma^2}{\rho} = e^{-\lambda}\{\nu'' + \nu'(\omega' - \tfrac{1}{2}\lambda')\}. \tag{14}$$

Further, if equations (9) and (10) are added,

$$\begin{aligned} 0 = {} & -e^{-\omega} + e^{-\lambda}(\tfrac{1}{2}\omega'\nu' + \tfrac{1}{2}\omega'^2 + \tfrac{1}{2}\omega'' + \tfrac{1}{2}\nu'' + \tfrac{1}{4}\nu'^2 - \tfrac{1}{4}\omega'\lambda' + \tfrac{1}{4}\omega'\nu' - \tfrac{1}{4}\nu'\lambda') \\ & + e^{-\nu}(-\dot{\omega}^2 + \tfrac{3}{4}\dot{\omega}\dot{\nu} - \tfrac{3}{2}\ddot{\omega} - \tfrac{1}{2}\ddot{\lambda} - \tfrac{1}{4}\dot{\lambda}^2 + \tfrac{1}{4}\dot{\lambda}\dot{\nu} - \tfrac{1}{4}\dot{\omega}\dot{\lambda}). \end{aligned} \tag{15}$$

Again, combining equations (9), (10) and (11),

$$\begin{aligned} 4\pi\rho - F^{14}F_{14} = {} & e^{-\lambda}[\tfrac{1}{4}\nu'^2 + \tfrac{1}{2}\{\nu'' + \nu'(\omega' - \tfrac{1}{2}\lambda')\}] \\ & + e^{-\nu}(-\ddot{\omega} - \tfrac{1}{2}\dot{\omega}^2 - \tfrac{1}{2}\ddot{\lambda} - \tfrac{1}{4}\dot{\lambda}^2 + \tfrac{1}{2}\dot{\omega}\dot{\nu} + \tfrac{1}{4}\dot{\lambda}\dot{\nu}). \end{aligned} \tag{16}$$

Using equations (6) and (14), equation (16) may be further simplified to

$$\left(1 - \frac{\sigma^2}{\rho^2}\right)(4\pi\rho + E^2) = -e^{-\nu}(\ddot{\omega} + \tfrac{1}{2}\dot{\omega}^2 + \tfrac{1}{2}\ddot{\lambda} + \tfrac{1}{4}\dot{\lambda}^2 - \tfrac{1}{2}\dot{\omega}\dot{\nu} - \tfrac{1}{4}\dot{\lambda}\dot{\nu}) \tag{17}$$

where E^2, the square of electric intensity, is defined as

$$E^2 = -F^{14}F_{14}.$$

Equation (17) may also be expressed with the help of equations (6) and (14) as

$$\left(\frac{\rho^2}{\sigma^2} - 1\right)\{\tfrac{1}{4}\nu'^2 + \tfrac{1}{2}\nu'' + \tfrac{1}{2}\nu'(\omega' - \tfrac{1}{2}\lambda')\} e^{-\lambda} = -e^{-\nu}\{\ddot{\omega} + \tfrac{1}{2}\dot{\omega}^2 + \tfrac{1}{2}\ddot{\lambda} + \tfrac{1}{4}\dot{\lambda}^2 - \tfrac{1}{2}\dot{\omega}\dot{\nu} - \tfrac{1}{4}\dot{\lambda}\dot{\nu}\}. \tag{18}$$

In the later discussion we shall consider the equations (12), (13), (15) and (18).

3. The case of shear-free expansion

The shear $q_{\alpha\beta}$ is defined by

$$q_{\alpha\beta} = \tfrac{1}{2}(v_{\alpha;\beta}+v_{\beta;\alpha})-\tfrac{1}{2}(\dot{v}_\alpha v_\beta+v_\alpha\dot{v}_\beta)-\tfrac{1}{3}(g_{\alpha\beta}-v_\alpha v_\beta) \tag{19}$$

with $\dot{v}_\alpha = v_{\alpha;\beta}v^\beta$ and $\theta = v^{\alpha}_{;\alpha}$. Equation (19) may be written out explicitly:

$$\begin{aligned} q_{\alpha\beta} = {} & \tfrac{1}{2}(v_{\alpha,\beta}+v_{\beta,\alpha}-2\Gamma^{\gamma}_{\alpha\beta}v_\gamma) \\ & -\tfrac{1}{2}\{v^\sigma v_\beta(v_{\alpha,\sigma}-\Gamma^{\gamma}_{\alpha\sigma}v_\gamma)+v^\sigma v_\alpha(v_{\beta,\sigma}-\Gamma^{\gamma}_{\beta\sigma}v_\gamma)\} \\ & -\tfrac{1}{3}\theta(g_{\alpha\beta}-v_\alpha v_\beta). \end{aligned}$$

In our coordinate system with the line element (1)

$$\theta = e^{-\nu/2}(\dot{\omega}+\tfrac{1}{2}\dot{\lambda})$$

and

$$v^\alpha = \frac{\delta^\alpha_4}{\sqrt{g_{44}}}.$$

With these conditions, one finds that $q_{\alpha\beta}$ vanishes except when $\alpha = \beta \neq 4$. If i, k run from 1 to 3, equation (19) reduces for

$$q_{ik} = e^{-\nu/2}\{\tfrac{1}{2}(\dot{g}_{ii})-\tfrac{1}{3}g_{ii}(\dot{\omega}+\tfrac{1}{2}\dot{\lambda})\}\delta_{ik}.$$

If shear is to vanish, this equation gives us $\dot{\omega} = \dot{\lambda}$, i.e. $\omega = \lambda+g(r)$ where g is a function of r only, so that $\omega' = \lambda'+g'$. Thus in the case of shear-free expansion (or contraction) equations (12), (13), (15) and (18) reduce to

$$-\dot{\lambda}'+\tfrac{1}{2}\dot{\lambda}\nu' = 0 \tag{20}$$

$$\dot{\nu}'+\tfrac{1}{2}\dot{\lambda}\nu' = 0 \tag{21}$$

$$\begin{aligned} & e^{-\lambda}\{\tfrac{1}{2}(\lambda''+\nu'')+(\tfrac{1}{4}\lambda'^2+\tfrac{1}{4}\nu'^2+\tfrac{1}{2}\lambda'\nu')+\tfrac{3}{4}g'(\lambda'+\nu')+(\tfrac{1}{2}g''+\tfrac{1}{2}g'^2-e^{-g})\} \\ & \quad +e^{-\nu}(-2\ddot{\lambda}-\tfrac{3}{2}\dot{\lambda}^2+\dot{\lambda}\dot{\nu}) = 0 \end{aligned} \tag{22}$$

$$\left(\frac{\rho^2}{\sigma^2}-1\right)(\tfrac{1}{2}\nu''+\tfrac{1}{4}\nu'^2+\tfrac{1}{4}\lambda'\nu'+\tfrac{1}{2}g'\nu') = e^{\lambda-\nu}(-\tfrac{3}{2}\ddot{\lambda}-\tfrac{3}{4}\dot{\lambda}^2+\tfrac{3}{4}\dot{\lambda}\dot{\nu}). \tag{23}$$

From equations (20) and (21)

$$\lambda = -\nu+f_1(t)+f_2(r)$$

but $f_1(t)$ can be removed by transformation of time coordinate, so that $\lambda = -\nu+f(r)$ or

$$e^\lambda = e^{-\nu}R_2 \tag{24}$$

where R_2 is a function of r only.

Again from equation (21) $\dot{\nu}'-\frac{1}{2}\nu'\dot{\nu} = 0$, so that on integration it gives

$$e^\nu = \frac{1}{(R_1+T)^2} \tag{25}$$

where R_1 is also a function of r alone and T is a function of t alone.

By equation (24)

$$e^\lambda = R_2(R_1+T)^2. \tag{26}$$

If we substitute equations (25) and (26) in equation (23) we obtain

$$\begin{aligned} & \left(\frac{\rho^2}{\sigma^2}-1\right)\left\{-\frac{R_1''}{R_1+T}-\frac{g'R_1'}{R_1+T}-\frac{R_1'R_2'}{2R_2(R_1+T)}+\frac{R_1'^2}{(R_1+T)^2}\right\} \\ & \quad = -3R_2(R_1+T)^3\left(\ddot{T}+\frac{\dot{T}^2}{R_1+T}\right). \end{aligned} \tag{27}$$

Let us now consider two cases, i.e. when $\sigma = \rho$ and when $\sigma \neq \rho$, separately.

When $\sigma = \rho$ the left-hand side of equation (27) vanishes and this gives

$$R_1 = -\frac{\dot{T}^2}{\ddot{T}} - T = \text{constant}.$$

As R_1 = constant, $R_1' = 0$ and $\nu' = 0$, i.e. ν is a function of time only. This leads to the uncharged case with $F_{14} = 0$ and $\sigma = 0$.

When $\sigma \neq \rho$, if equation (27) is successively differentiated suitably with respect to r or t, we finally reach the condition that either $\dot{T} = 0$ or R_1 = constant.

The first condition leads to the static case with $\dot{\nu} = \dot{\lambda} = 0$. This corresponds to the equilibrium charge dust distribution as was pointed out by Bonner (1960).

The second condition again leads to the uncharged case with $F_{14} = 0$ and $\sigma = 0$. This corresponds to the same condition, as was pointed out by Oppenheimer and Snyder.

Thus we arrive at the theorem that an expansion (or contraction) of a spherically symmetric charged dust cannot be isotropic (i.e. shear free).

4. Perturbation of the equilibrium distribution

For a charged dust distribution in equilibrium one has $\sigma = \pm\rho$ (De and Raychaudhury 1968). To consider the perturbation of the equilibrium distribution, we start with the defining equation

$$v^{\alpha}_{;\alpha\beta} - v^{\alpha}_{;\beta\alpha} = R_{\alpha\beta} v^{\alpha}.$$

From this equation one obtains (Ehlers 1961)

$$R_{\alpha\beta} v^{\alpha} v^{\beta} = \theta_{;\alpha} v^{\alpha} - \Pi^{\alpha}_{;\alpha} + 2(q^2 - \omega^2) + \tfrac{1}{3}\theta^2 \tag{28}$$

where θ, q and ω are the expansion, shear and rotation defined as

$$\theta = v^{\alpha}_{;\alpha}$$

$$q = (\tfrac{1}{2} q_{\alpha\beta} q^{\alpha\beta})^{1/2}$$

and $q_{\alpha\beta}$ is given by equation (12)

$$\omega = (\tfrac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta})^{1/2}$$

and

$$\omega_{\alpha\beta} = \tfrac{1}{2}(v_{\alpha;\beta} - v_{\beta;\alpha}) - \tfrac{1}{2}(\dot{v}_{\alpha} v_{\beta} - v_{\alpha}\dot{v}_{\beta}).$$

Further

$$\Pi^{\alpha} = v^{\alpha}_{;\beta} v^{\beta} \tag{29}$$

so that in the case of charged dust

$$\Pi^{\alpha} = -\frac{\sigma}{\rho} F^{\alpha}_{.\beta} v^{\beta} \tag{30}$$

or

$$\Pi^{\alpha}_{;\alpha} = -\left(\frac{\sigma}{\rho}\right)_{,\alpha} (F^{\alpha\beta} v_{\beta}) + 4\pi \frac{\sigma^2}{\rho} + \frac{\sigma}{\rho}(F^{\alpha\beta}\omega_{\alpha\beta}) - \left(\frac{\sigma}{\rho}\right)^2 (F^{\alpha\beta} v_{\beta})(F_{\alpha\nu} v^{\nu}). \tag{31}$$

We now define local electric and magnetic fields as

$$E^{\alpha} = F^{\alpha\nu} v_{\nu}$$

$$H^{\alpha} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\mu} F_{\beta\gamma} v_{\mu}$$

then by equations (2), (28) and (31) we obtain

$$4\pi\rho\left(1 - \frac{\sigma^2}{\rho^2}\right) + E^2\left(1 - \frac{\sigma^2}{\rho^2}\right) + H^2$$

$$= -\tfrac{1}{3}\theta^2 - \theta_{;\alpha} v^{\alpha} + \frac{\sigma}{\rho}(F^{\alpha\beta}\omega_{\alpha\beta}) - 2(q^2 - \omega^2) - \left(\frac{\sigma}{\rho}\right)_{,\alpha} (F^{\alpha\beta} v_{\beta}). \tag{32}$$

For the spherically symmetric case, the rotation ω and the magnetic field H vanish, so that for the case $\sigma = \pm\rho$, we have $\frac{1}{3}\theta^2 + \theta_{;\alpha}v^\alpha = -2q^2$, or as $\theta = v^\alpha_{;\alpha}$, writing $G^3 = \sqrt{-g}$ we have $\ddot{G}/G = -2q^2$, so that $\ddot{G}$ is essentially negative. Hence, if as a result of a small perturbation the system starts expanding (or contracting), the expansion (or contraction) will be decelerated (or accelerated).

Thus eventually, if the perturbation were sufficiently small, the expansion would be reversed and the collapse would proceed to a singularity in a finite time. Hence we arrive at the second theorem: that a static charge is unstable to perturbations, which do not destroy the spherical symmetry, and would eventually collapse.

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